Geometric Characterization of True Quantum Decoherence

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Surprisingly often decoherence is due to classical fluctuations of ambient fields and may thus be described in terms of random unitary (RU) dynamics. However, there are decoherence channels where such a representation cannot exist. Based on a simple and intuitive geometric measure for the distance of an extremal channel to the convex set of RU channels we are able to characterize the set of *true quantum* phase-damping channels. Remarkably, using the Caley-Menger determinant, our measure may be assessed directly from the matrix representation of the channel. We find that the channel of maximum quantumness is closely related to a symmetric, informationally-complete positive operator-valued measure (SIC-POVM) on the environment. Our findings are in line with numerical results based on the entanglement of assistance.

I. INTRODUCTION

In quantum mechanics, any two distinct states of a quantum system may be coherently superposed to yield a novel state. In addition to the occupation probabilities of the individual states, such a superposition is characterized by the coherences which describe its interference potential. The theory of quantum information processing exploits the superposition principle in the design of algorithms whose efficiencies by far exceed conventional, classical schemes [1]. In practice, however, the coherences are susceptible to decay—a process usually termed decoherence [2–4]. Sometimes it is possible to single out a certain basis of robust states which is insensitive to the given decoherence dynamics (often a simple matter of time scale [5, 6]). With respect to this preferred basis all populations remain unchanged and only the coherences are subject to decay. One then also speaks of pure decoherence or phase damping (a.k.a. dephasing).

In experiment, phase damping is often due to classical fluctuations of ambient fields (sometimes also called "random external fields" [1, 7, 8]). These fluctuations have for example been identified as the main source of decoherence in ion trap quantum computers, where instabilities are present both in the trapping fields and in the laser addressing the individual ions [9, 10]. Formally, decoherence is then of random unitary (RU) nature, i.e., it may be represented in terms of a stochastic ensemble of unitary dynamics. Despite the practical relevance of this classical, ensemble-based approach, the common understanding of decoherence is based on the language of open quantum systems. Here, the loss of coherences is a direct consequence of growing correlations between the system and its quantum environment. Note that while these correlations may well turn out to involve quantum entanglement, surprisingly often the system may have decohered completely while still being separable from the environment [11, 12].

While RU dynamics may certainly account for a wide range of experimentally observed phase damping, there are examples of *true quantum* decoherence, where such a RU representation is impossible (i.e., the decoherence cannot be attributed to classical fluctuations) [13–15]. The existence of such quantum channels has sparked the research in quantum information processing community on the asymptotic version of quantum Birkhoff's theorem [16] which was recently solved [17]. Despite some mentionable effort there is no known simple criterion allowing to decide whether or not a given dynamics is RU [18–21]. The purpose of the present article is to provide a characterization of the set of true quantum phase-damping dynamics, i.e., dynamics which may not be explained in terms of fluctuating fields. Our analysis is based on a simple and intuitive geometric measure—a volume—for the distance of a given decoherence dynamics to the convex set of RU dynamics. We find strong numerical evidence that our results are also valid for another measure of quantumness based on the entanglement of assistance.

II. PRELIMINARIES

In this work we work with finite dimensional Hilbert spaces, namely $\mathcal{H} = \mathbb{C}_N$. The quantum channel formalism provides a viable tool to account for a wide variety of open quantum system dynamics [1]. It relies on completely positive, trace-preserving maps, mapping the initial state of the system of interest onto its time-evolved image, $\mathcal{E} : \varrho \mapsto \varrho'$. Any such map may be written using the Kraus representation

$$\varrho' = \sum_i K_i \varrho K_i^\dagger, \tag{1}$$

where the trace-preserving character implies $\sum_i K_i^\dagger K_i = \mathbb{1}$. With r we denote the Kraus rank of a channel, giving the number of Kraus operators in (1) with $\{K_i\}$ linear independent. A channel is called *unital* if it leaves the completely mixed state unchanged. Then, the Kraus operators additionally obey $\sum_i K_i K_i^\dagger = \mathbb{1}$ [27]. We further assume that the input and output state spaces of the quantum channel $\mathcal{E}$ have the same dimensionality.

A prime example is given by a RU channel, where the K_i may be chosen to be unitary up to a trivial (positive)

factor, that is,

$$\varrho' = \sum_i p_i U_i \varrho U_i^\dagger \quad \left(p_i \geq 0, \sum_i p_i = 1\right).$$

In addition to the already mentioned experimental significance, RU channels are frequently studied due to their analytical accessability [22–24].

The case of pure decoherence stands out due to the existence of a basis $|n\rangle$ of robust states, suggesting the channel may be written in the form

$$\varrho'_{mn} = \langle a_n|a_m\rangle \varrho_{mn}, \tag{2}$$

with $\{|a_n\rangle\} \subset \mathbb{C}^r$ being a set of normalized complex vectors—the *dynamical vectors*. It is then sometimes convenient to introduce the matrix D with $D_{mn} = \langle a_n|a_m\rangle$, such that the channel may be expressed in terms of the Hadamard product, $\varrho' = D \star \varrho$, denoting the entry-wise product of matrices of the same size [26].

A. True Quantum Phase-Damping

It is known that the convex set of unital channels contains non-unitary, extremal channels [13, 14]. These channels undeniably represent examples of *true* quantum decoherence, i.e., decoherence which may not be understood in terms of RU dynamics. A phase-damping channel is known to be extremal if the projectors onto the dynamical vectors, $\Pi_n = |a_n\rangle\langle a_n|$, form a (possibly overcomplete) operator basis on the set of $r \times r$ matrices $\mathcal{M}_r$ [13]. This extremality criterion has a very intuitive, geometric interpretation in terms of the Bloch representation. Recall that the Bloch vector $\vec{b}_n \in \mathbb{R}^{r^2-1}$ corresponding to projector Π_n may be assigned via

$$\Pi_n = \frac{1}{2}\left(\frac{2}{r}\mathbb{1}_r + \vec{b}_n \cdot \vec{\sigma}\right),$$

where $\vec{\sigma} = (\sigma_1, \ldots, \sigma_{r^2-1})$ is the vector of a set of orthogonal traceless generators of the SU(r) with $\mathrm{tr}\sigma_i\sigma_j = 2\delta_{ij}$ [27]. A simple argument shows that extremality is given as soon as the Bloch vectors span a non-zero volume in $\mathbb{R}^{r^2-1}$ [15]. While $r > 1$ is certainly necessary for the channel to be non-unitary, extremality requires $r^2 \leq N$. The smallest possible dimension allowing for a non-RU channel is thus given by $N = 4$ (a system of two qubits).

B. Measures of Quantumness

Besides plain identification of the non-classical nature we are also interested in a more quantitative estimation of the *quantumness* of a given channel (i.e., its distance to the set of RU channels). The first approach (proposed in [19]) is based on the entanglement of assistance

$$E_A(\varrho) := \max_{\{p_i, |\psi_i\rangle\}} \left\{ \sum_i p_i E(\psi_i) \;:\; \sum_i p_i |\psi_i\rangle\langle\psi_i| = \varrho \right\}.$$

It gives the maximum average entanglement entropy E over all pure-state decompositions of a given state. Recall that via the Jamiolkowski isomorphism [28] we can assign to every channel $\mathcal{E}$ a state $\varrho_{\mathcal{E}}$ by applying $\mathcal{E}$ to one half of the maximally entangled state $\sum_n |nn\rangle / \sqrt{N}$:

$$\varrho_{\mathcal{E}} = \frac{1}{N}(\mathcal{E} \otimes \mathbb{1}) \sum_{m,n} |mm\rangle\langle nn|.$$

Since entanglement is invulnerable to local unitary transformations, the resulting state is a convex mixture of maximally entangled pure states if and only if the channel is RU. Trivially, the state's entanglement of assistance then yields the maximal amount of $E_A(\varrho_{\mathcal{E}}) = \log_2 N$, while any smaller value clearly indicates that the channel is of true quantum type. As a minor adaption we choose to additionally normalize the result, defining the "quantumness of assistance" as

$$Q_A(\mathcal{E}) = 1 - \frac{E_A(\varrho_{\mathcal{E}})}{\log_2 N}.$$

Now we have that $0 \leq Q_A \leq 1$ and $Q_A = 0$ if and only if the channel is RU.

As previously discussed, the Bloch representation offers a geometric approach to identify extremality. Our previous results furthermore suggest that the volume spanned by the relative states gives a good estimation of the quantumness of the channel [15]. A very elegant way to directly assess the Bloch volume may be obtained using the Caley-Menger determinant (see Appendix A). For simplicity let us first consider the case $r^2 = N$. Then, the N Bloch vectors form a $N-1$ simplex in $\mathbb{R}^{N-1}$ spanning a volume given by

$$V_B^2 = \frac{(-1)^N}{2^{N-1}((N-1)!)^2} \begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & 4\,(\mathrm{id} - D \star D^*) & \\ 1 & & & \end{vmatrix}.$$

In the case $r^2 < N$, the direct calculation of the determinant will yield zero volume at all times. Rather, one has to check the corresponding determinant for all r^2-dimensional submatrices of D obtained by discarding $N - r^2$ rows and columns of the same indices because extremal channels with rank $r^2 < N$ are directly related to extremal channels with $r^2 = \tilde{N} < N$.

III. GEOMETRIC CHARACTERIZATION

The geometric character of the Bloch volume may now be exploited in order to characterize the set of two-qubit phase-damping channels. As an additional parameter in our investigation we consider the purity, which for a quantum state is defined as the trace of the squared density matrix, $P(\varrho) := \mathrm{tr}\left\{\varrho^2\right\}$. In the case of a quantum channel D, we may evaluate the purity of the corresponding Jamiolkowski state, $P(\varrho_D)$. Note that it is

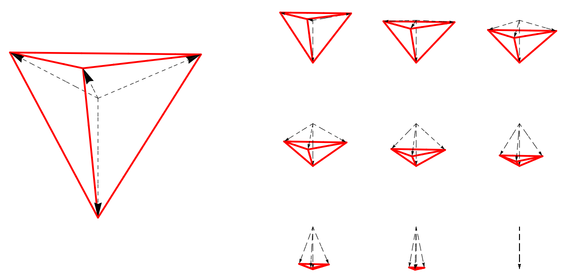

FIG. 1. (Color online) Schematic Plot of the Bloch vectors (black, dashed arrows) corresponding to the "maximum quantum" channel (big) and to a selection of MCMQ (small) spanning tetrahedra (red, solid lines) of decreasing volume.

straightforward to see that a unital channel represents unitary dynamics if and only if $P(\varrho_D) = 1$. On the other hand, the minimum value of $P(\varrho_D) = 1/N$ is only possible if $D_{mn} = 0$ for all $m, n = 1, \ldots, N$ with $m \neq n$. Then, however, *any* initial state is mapped onto a state with diagonal density matrix. We may call this channel the *completely decohering channel*, D_{cd}. In this vein, the purity of a channel may thus be used as an indication of the effect of the channel on any arbitrary initial state. If it is close to 1, the effect of the channel on the purity of an arbitrary state is probably small. If it approaches a value of $1/N$, however, it's impact in terms of purity is possibly large.

From geometric considerations it is quite trivial to arrive at the phase-damping channel of maximum Bloch volume: the corresponding Bloch vectors span a regular tetrahedron inside the Bloch sphere. A possible choice is given by the four vectors

$$\vec{b}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \vec{b}_2 = \begin{pmatrix} \sin\alpha \\ 0 \\ \cos\alpha \end{pmatrix},$$
$$\vec{b}_3 = \begin{pmatrix} \sin\alpha\cos\frac{2\pi}{3} \\ \sin\alpha\sin\frac{2\pi}{3} \\ \cos\alpha \end{pmatrix}, \quad \vec{b}_4 = \begin{pmatrix} \sin\alpha\cos\frac{2\pi}{3} \\ -\sin\alpha\sin\frac{2\pi}{3} \\ \cos\alpha \end{pmatrix}, \tag{3}$$

where $\alpha = \arccos\left(-\frac{1}{3}\right)$ denotes the so-called tetrahedral angle. With $x := \sqrt{\frac{1+\cos\alpha}{2}} = \sqrt{\frac{1}{3}} \approx 0.57735$, the corresponding phase damping channel has the matrix representation

$$D_\Delta = \begin{pmatrix} 1 & x & x & x \\ x & 1 & ix & -ix \\ x & -ix & 1 & ix \\ x & ix & -ix & 1 \end{pmatrix}. \tag{4}$$

Note that this symmetric placement of the Bloch vectors is known from the concept of SIC-POVMs. In $\mathbb{C}^N$, a SIC-POVM is defined as a set of N^2 normalized vectors $|\psi_i\rangle$ with $|\langle\psi_i|\psi_j\rangle|^2 = \frac{1}{N+1}, \quad i \neq j$ [25].

As previously discussed, a unitary channel has purity $P = 1$. The channel with maximum volume, Eq. (4), on the other hand, has purity $P(\varrho_{D_\Delta}) = 1/2$. What can be said about channels with intermediate values of purity? Certainly, there must exist channels maximizing the volume for a given purity. When expressed in terms of the Bloch vectors of the dynamical vectors, the Purity of the channel D is given by $P(\varrho_D) = \frac{1}{2}(1 + |\vec{b}_S|^2)$, where $\vec{b}_S = (\vec{b}_1 + \ldots + \vec{b}_4)/4$ denotes the barycentre of the Bloch vectors. Of all tetrahedra whose barycenters lie equidistant from the origin we thus aim to identify the one spanning the maximum volume. We find this maximum numerically: it is attained for the Bloch vectors in Eq. (3), yet with $\alpha \in [0, \arccos(-1/3)]$. For $\alpha = 0$ the volume is zero, the purity equals one, and the corresponding channel is unitary. For $\alpha = \arccos(-1/3)$ we get the channel with maximum volume. The values in between give the *mixed channels with maximum quantumness* (MCMQ). The picture that comes to mind is the process of closing an umbrella: while one of the Bloch vectors remains fixed, the three remaining ones move towards the first just like the metal frame carrying the fabric (Fig. 1).

IV. COMPARISON WITH THE QUANTUMNESS OF ASSISTANCE

In order to check the validity of our findings with respect to the quantumness of assistance we first compare the two measures directly. For this, we study a set of randomly generated two-qubit phase-damping channels. The random generation is based on representation (2): we draw a set of vectors $|a_n\rangle$, $n = 1, \ldots, 4$, which are equally distributed on the unit sphere in $\mathbb{C}^r$. The channel is then obtained as the Gram matrix $(D_{mn}) = (\langle a_m|a_n\rangle)$.

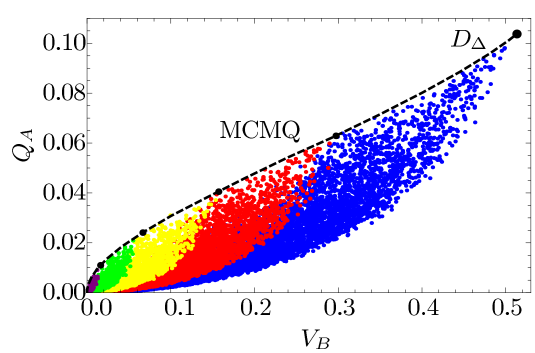

FIG. 2. (Color online) Quantumness Q_A vs. Bloch volume V_B for a set of random phase-damping channels of rank 2. The color coding indicates the purity of the channel in the following way: blue ($0.5 \leq P < 0.6$), red ($0.6 \leq P < 0.7$), yellow ($0.7 \leq P < 0.8$), green ($0.8 \leq P < 0.9$), purple ($0.9 \leq P < 1$). The dashed line corresponds to the one-parameter set of MCMQ, the channel with maximum Bloch volume D_Δ is indicated with a black dot, as are the MCMQ channels with $P = 0.6$, 0.7, 0.8, and 0.9 (from right to left).

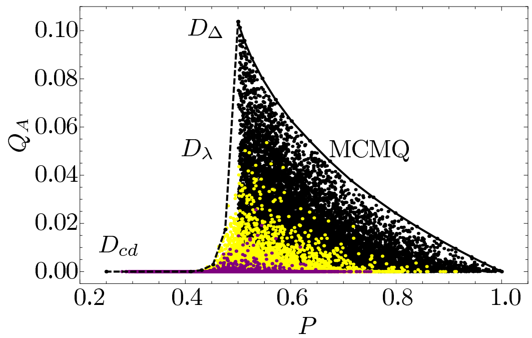

FIG. 3. (Color online) Quantumness in terms of Purity for a set of random phase-damping channels. The color coding indicates the rank of the respective channels: rank 2 (black), rank 3 (yellow), and rank 4 (purple). For further details see text.

At first, only channels or rank $r = 2$ are studied (allowing for the channels to be extremal in the set of unital channels). To each channel we calculate Bloch volume V_B and quantumness Q_A, as well as the purity P.

We observe that, depending on the purity of the channel, there exist certain bounds to the quantumness as well as to the Bloch volume: the lower the purity of the channel, the higher the accessible quantumness and volume. In Fig. 2 this is highlighted using a color scheme to single out specific purity intervals. Certainly, there is no one-to-one correspondence between the two measures; yet, the correlation is evident. It is quite apparent that towards the channel of maximum Bloch volume the two measures converge. This is also true for any of the purity intervals specified. It is thus maybe not too surprising that the values obtained for the class of MCMQ represents an upper bound to the accessible quantumness of assistance in terms of fixed Bloch volume (dashed line in Fig. 2). These findings strongly suggest that the MCMQ also maximize the quantumness of assistance.

Our findings are best visualized in the Quantumness-Purity-Plane (Fig. 3). Here, also channels of rank 3 and 4 are included. For a total number of 70000 random channels we find no single violation of the upper bound represented by the MCMQ (solid line). In addition, we observe that below a purity of 0.5 there are only few channels with considerable quantumness. In order to get a feeling for this behavior we look at channels that are defined as convex mixture of the channel with maximum volume, D_Δ, and the completely decohering channel, D_{cd}: $D_\lambda = (1-\lambda)D_\Delta + \lambda D_{cd}$. Note that it is easy to see that D_{cd} belongs to the set of RU channels: its RU decomposition is given by $(\mathbb{1}\otimes\mathbb{1})/4, (\sigma_z\otimes\mathbb{1})/4, (\mathbb{1}\otimes\sigma_z)/4, (\sigma_z\otimes\sigma_z)/4$. For this one-parameter class of channels we numerically estimate the quantumness. We find that the corresponding quantumness rapidly decays to zero for increasing λ (see the dashed line in Fig. 3). All sampled channels are RU already for $\lambda > 0.2$. This value is considerably lower than the bound $\lambda^* = \frac{14}{15}$ which makes *any* channel RU [30].

V. SUMMARY

Based on a simple and intuitive geometric measure—a volume—we are able to characterize the set of true quantum phase-damping channels acting on two qubits. We identify the channel with the maximum distance to the set of RU channels, which is directly linked to the concept of SIC-POVM. In this context our results imply that the maximally non-classical phase-damping channel corresponds to a set of rank-one measurements which is best equipped for distinguishing quantum states on the environment—a point certainly worth further study. Our findings are in remarkable agreement with numerical results based on the entanglement of assistance of the Jamiolkowski state of the channel.

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Appendix A: Assessing the Bloch volume with the Caley-Menger determinant

The volume of a $N-1$ simplex spanned by the vectors $\vec{x}_n = (x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(N-1)}) \in \mathbb{R}^{N-1}, n = 1,\ldots,N$, may be evaluated using the so-called Caley-Menger determinant [29]. With $s_{mn} = \sqrt{(\vec{x}_m - \vec{x}_n)\cdot(\vec{x}_m - \vec{x}_n)}$ denoting the distance between vertex m and n it is defined as

$$\det(A_N) = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & s_{1,2}^2 & \cdots & s_{1,N}^2 \\ 1 & s_{1,2}^2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & s_{N-1,N}^2 \\ 1 & s_{1,N}^2 & \cdots & s_{N-1,N}^2 & 0 \end{vmatrix}. \quad \text{(A1)}$$

The volume of the simplex is then given by

$$\mathrm{Vol}^2 = \frac{(-1)^N}{2^{N-1}((N-1)!)^2}\det(A_N). \quad \text{(A2)}$$

A phase-damping channel D in dimension N is defined via $D_{mn} = \langle a_n|a_m\rangle, \quad m,n = 1,\ldots,N$. For a channel of rank r we may assign to each dynamical vector $|a_n\rangle \in \mathbb{C}^r$ a Bloch vector $\vec{b}_n \in \mathbb{R}^{r^2-1}$. The mutual distance between any of these Bloch vectors equates to $s_{m,n}^2 = 4\left(1-\frac{1}{r}\right) - 2\,\vec{b}_m\cdot\vec{b}_n$, while the matrix elements D_{mn} may be expressed as $|D_{mn}|^2 = |\langle a_n|a_m\rangle|^2 = \frac{1}{r} + \frac{1}{2}\vec{b}_n\cdot\vec{b}_m$. Put together, it is straightforward to see that $s_{m,n}^2 = 4\left(1-|D_{mn}|^2\right)$, so that we arrive at the equivalence

$$V_B^2 = \frac{(-1)^N}{2^{N-1}((N-1)!)^2}\begin{vmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & 4\,(\mathrm{id} - D\star D^*) & \\ 1 & & & \end{vmatrix}. \text{(A3)}$$

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